Lecture 24

Tuesday, October 26, 2021 9:53 PM

* Prayer

* Spiritual thought

* About a homework problem:

$$y = \sum a_n x^n = c_1 y_1 + c_2 y_2.$$

y_1 is the solution to $\begin{cases} y'' - xy' - y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases} \longrightarrow y_1 = \sum b_n x^n$

y_2 is the solution to $\begin{cases} y'' - xy' - y = 0 \\ y(0) = 0 \\ y'(0) = 1 \end{cases} \longrightarrow y_2 = \sum c_n x^n$

$y = c_1 y_1 + c_2 y_2$ where $c_1 = y(0) = a_0$, $c_2 = y'(0) = a_1$.

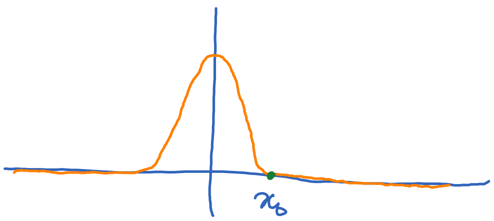
A drawback of the power-series method is that the solution found has to be an analytic function.

If f is analytic at x_0 then f is infinitely differentiable at x_0.

If f is not differentiable infinitely many times at x_0 then it is not analytic at x_0.

However, being infinitely differentiable is not a guarantee for a function to be analytic.

Ex:



$f^{(n)}(x_0) = 0 \quad \forall n.$

If f is analytic at x_0 then

$f(x) = \sum a_n x^n$ on some interval $(x_0 - r, x_0 + r)$.

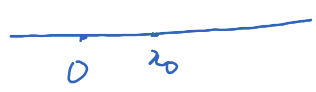
Taylor: $\quad a_n = \dfrac{f^{(n)}(x_0)}{n!} = 0 \quad \forall n$

$\rightsquigarrow \; f(x) = 0$ for all $x \in (x_0 - r, x_0 + r) \; \rightsquigarrow$ Contradiction!

<u>Question</u>: How do we know if a function is analytic at a given point?

An elementary function (polynomial, fractional, exponential, logarithmic, trigonometric and the combinations) is analytic at any point where it is continuous.

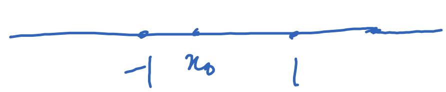
Ex: $\quad \frac{1}{x}$ is analytic at any $x_0 \neq 0$.



$\frac{1}{x} = \sum a_n (x - x_0)^n$

radius of conv. $= |x_0|$.

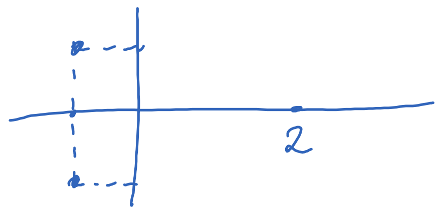
Ex: $\quad \ln(x^2 - 1)$ is analytic at any $x_0 < -1$ or > 1.



radius of conv. $= \min\{|x_0 - 1|, |x_0 + 1|\}$

Ex $\dfrac{x+2}{x^2+2x+2}$ is analytic at any $x_0 \in \mathbb{R}$.

Singularity: $x^2+2x+2=0 \rightsquigarrow x=-1\pm i$



Radius of conv. is

$$\min\{|x_0-(-1-i)|,\ |x_0-(-1+i)|\}$$
$$=|x_0-(-1+i)| = |x_0+1-i|$$
$$=\sqrt{(x_0+1)^2+1}$$

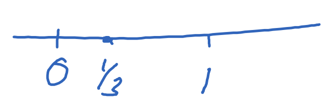
Question: when does the power series method work?

$$y^{(n)}+p_1(x)y^{(n-1)}+\cdots+p_n(x)y=g(x),\quad y(x_0)=y_0,\ \ldots,\ y^{(n-1)}(x_0)=y_0^{(n-1)}.\quad (*)$$

If $p_1, p_2, \ldots, p_n, g$ are analytic at x_0 then $(*)$ has a unique analytic solution $y=\sum a_n(x-x_0)^n$.

Ex: $x\ln(1-x)\,y'+e^x y=\sqrt{x}$, $\quad y(\tfrac{1}{3})=1$.

$$\rightsquigarrow\ y'+\underbrace{\frac{e^x}{x\ln(1-x)}}_{\substack{\text{analytic}\\ \text{at } x_0=1/3}}y=\underbrace{\frac{\sqrt{x}}{x\ln(1-x)}}_{\substack{\text{analytic}\\ \text{at } x_0=1/3}}$$



$y=\sum a_n(x-\tfrac{1}{3})^n \rightsquigarrow$ radius of convergence $\geq \tfrac{1}{3}$.